On Maximal Ideal Cycles for $2$-Dimensional Normal Double Points

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SUMMARY: In this paper we study the maximal ideal cycle on the minimal good resolution of a normal double point $(X, o)$ defined by $z^2 = f(x, y)$ over the complex number field. We compare it to the fundamental cycle on the minimal resolution of $(X, o)$. By using an argument of covering surface, we show that the ratio of coefficients of such both cycles on any exceptional component is always $1$ or $2$.

1. INTRODUCTION AND PRELIMINARIES.

Let $\pi: (\hat{X}, A) \rightarrow (X, o)$ be a resolution of a normal complex surface singularity, where $\pi^{-1}(o) = A = \bigsqcup A_i$ is the irreducible decomposition of the exceptional set $A$. The fundamental cycle $Z_A$ on $A$ is a unique smallest positive cycle $\sum z_i A_i$ ($z_i \in \mathbb{Z}$) which satisfy $Z_A \cdot A_i \leq 0$ for any $i$ (11). Further we denote by $m$ the maximal ideal cycle of $\mathcal{O}_{X,o}$. The maximal ideal cycle $M_A$ on $A$ is a unique positive cycle $\sum m_i A_i$ ([2], [3]), where $m_i = \min_i v_{A_i}(\pi^* f) \in \mathbb{N}$ and $v_{A_i}$ is the valuation determined by $A_i$ (i.e., the vanishing order on $A_i$). Then we have $Z_A \leq M_A$.

(1.1) $0 < -Z_A^2 \leq -M_A^2 \leq \text{mult}(\mathcal{O}_{X,o})$ and $m \mathcal{O}_{X,o} \subseteq \mathcal{O}(-M_A)$, where $\text{mult}(\mathcal{O}_{X,o})$ is the multiplicity of $\mathcal{O}_{X,o}$ ([2]).

Every normal double point (i.e., multiplicity $= 2$) is defined by $z^2 = f(x, y)$ (cf. [1],[4]), where $f \in \mathbb{C}[x, y]$. For normal double points, D.J. Dixon [5] compared the fundamental cycle $Z$ and the maximal ideal cycle $M$. He proved that if $\text{ord}(f)$ is even, $M$ equals $Z$ for any resolution. Further he proved that if $f$ is irreducible and $\text{ord}(f)$ is odd, $M$ equals $Z$ on the minimal resolution. In this paper we also compare $M$ and $Z$, when $\text{ord}(f)$ is odd but $f$ is not necessarily irreducible.

In section 2 we prove our main results. Let $(\hat{X}, A)$ be the minimal resolution or the minimal good resolution of a normal double point $(X,o)$. Let $M_A$ and $Z_A$ be the maximal ideal cycle and the fundamental cycle respectively. We prove that there is a decomposition $A = A(1) \cup A(2)$ such that $A(i)$ is a connected 1-dimensional subvariety for $i=1,2$ and $M|_{A(1)} = 2Z|_{A(1)}$ and $M|_{A(2)} = Z|_{A(1)}$, where $M|_{A(1)}$ is the restriction of $M$ onto $A(1)$ and so on. This result can be also proved by an entirely different method due to Karras (see [6]). However, our proof of this paper is important from the technical point of view.

In general, the maximal ideal cycle for a resolution is not determined from the w.d. graph (= weighted dual graph) of the exceptional set. For example, w.d. graphs for $z^2 = x^6 + y^8$ and $z^2 = y(x^4 + y^4)$ are identical. It is given by the following configuration:

$$
\begin{array}{ccc}
A_1 & A_2 & A_3 \\
\circ & \circ & \circ \\
\end{array}
$$

However the maximal ideal cycles for them are different and they are given by $A_1 + A_2 + A_3$ and $2A_1 + 2A_2 + A_3$ respectively (see p.48 in [7]).

From now on we prepare some facts and terminologies for normal double points according to [4]. Let $(X, o)$ be a normal double point defined by $z^2 = f(x, y)$, where $f \in \mathbb{C}[x, y]$ with $\text{ord}(f) \geq 2$. Then $f$ does not contain any multiple factor. Let $C = \{(x, y) = 0 \subseteq \mathbb{C}^2 \}$ be a curve singularity. We obtain an embedded resolution $C^2 \xrightarrow{\sigma_1} V_1 \xrightarrow{\sigma_2} ... \xrightarrow{\sigma_k} V = V_e$ of $(C, o)$ by an iteration of blowing-ups (i.e., the strict
transform of $C$ is a simple normal crossing divisor in $V_1$. By taking a fiber product we have the following diagram:

$$
\begin{align*}
\begin{array}{c}
\xymatrix{ 
C^2 \ar[r]^{-\sigma_{1}, \cdots, \sigma_{t}} & V \times C^1 \\
\ar@{^(->}[u] \ar[u]_p \ar[r] & \ar@{^(->}[u] \ar[u]_{\pi} \ar[r] & \ar@{^(->}[u] \ar[u]_\pi \\
V_0 = C^2 \ar[r]^{\sigma_{1}, \cdots, \sigma_{t}} & V_1 \ar[r] & \cdots \ar[r] & V = V_{t+1} 
\end{array}
\end{align*}
\tag{1.1}
$$

where $\sigma = \sigma_{1}, \cdots, \sigma_{t}$ and $p$ is a double covering induced by the projection $C^2 \rightarrow C^1$ $(x, y, z) \mapsto (x, y)$, and then $\pi$ is a double covering map, too. The map $\sigma$ is a birational morphism. Let $\tilde{E}$ be the strict transform of the $(1)$-exceptional curve of $\sigma$, onto $V$ by $\sigma$, $\cdots, \sigma_{i+1}$ $(i = 1, 2, \cdots, s)$ and $\phi$ is the normalization of $V$. Let $C = \bigsqcup C_j$ be the irreducible decomposition and let $\tilde{C}_j$ be the strict transform by $\sigma(j = 1, 2, \cdots, r)$. Let $E = \pi^{-1}(\tilde{E})$ and $E_i = \pi^{-1}(\tilde{C}_i)$, so $E_i$ is not always irreducible. Further we put $C = \bigsqcup C_j$.

Definition 1.1. We call $\tilde{E}_i$ an $(f)$-odd (resp. $(f)$-even) curve if the vanishing order $\nu_{\tilde{E}_i}(f \circ \sigma)$ is odd (resp. even). Further, since $\nu_{\tilde{C}_j}(f \circ \sigma) = 1$ for any $\tilde{C}_j$, we call $\tilde{C}_j$ an $(f)$-odd curve.

Please refer [5] about the relations between self-intersections of $E_i$ and $\tilde{E}_i$, and also coeff$_{\tilde{E}_i}$ $M$ and coeff$_{\tilde{E}_i}Z$.

Notations and terminologies. Let $\pi : (\tilde{X}, A) \rightarrow (X, o)$ be a resolution of a normal surface singularity, where $\pi^{-1}(o) = A = \bigsqcup A_i$ is the irreducible decomposition of the exceptional set $A$. For a cycle $D = \sum_i d_iA_i$ on the exceptional set $A$, let's coeff$_{\tilde{E}_i}$($D$) denote the coefficient $d_i$ of $D$ on $A_i$. For an exceptional subset $A' \subset A$, $D|_{A'}$ is the restriction of $D$ onto $A'$. Namely $D|_A = \sum_{A_i \subset A} d_iA_i$. For $f \in C(x, y)$, let ord $(f)$ be the order of zeros of $f$. Further, $(-1)$-curve means a projective line whose self-intersection number equals $-1$.

2. MAXIMAL IDEAL CYCLES AND FUNDAMENTAL CYCLES.

In this section we study the relation between the maximal and the fundamental cycles for a normal double point $(X, o) = \{z^2 = f(x, y)\}$, where $f \in C(x, y)$. If ord $(f)$ is even, then $Z^2 = -2$ (this fact was included in the proof of Theorem 1 in [4]). Therefore, in this section, we only consider the case where ord $(f)$ is odd. Let $\tilde{M}$ be the maximal ideal cycle on the minimal resolution. Since $0 \leq -Z^2 \leq -\tilde{M}^2$, we classify such normal double points into three types as follows:

- type $(2, 2)$ : $Z^2 = -2$
- type $(1, 2)$ : $Z^2 = -1$ and $\tilde{M}^2 = -2$
- type $(1, 1)$ : $\tilde{M}^2 = -1$.

We can see that if $(X, o)$ is type $(2, 2)$ (resp. $(1, 1)$), then $\tilde{M}^2 = -2$ (resp. $Z^2 = -1$). For example, let $(X, o) = \{z^2 = x^2 + y^2\}$. If $2 \leq n \leq 5$ (resp. $6 \leq n$), then $(X, o)$ is of type $(2, 2)$ (resp. $(1, 1)$). Further the singularity $z^2 = x^2 + y^2$ described in section 1 is of type $(1, 2)$. For any double point with $Z^2 = -2$, we have $M = Z$ for any resolution. Hence we are only concerned for the case of $Z^2 = -1$ in this section.

In this section we prove that the ratio of coefficients of $M$ and $Z$ on any exceptional curve in any covering resolution is always one or two. Our main result in this section is Theorem 2.4.

In the following of this section, we assume that $Z^2 = -1$ (so ord $(f)$ is odd) and $(\tilde{X}, \tilde{E})$ is a covering resolution over $(X, E)$ as in (1.1), and let $\tilde{E} = \bigsqcup \tilde{E}_i$ and $E = \bigsqcup E_i$. We note that $E_i$ is not necessarily irreducible. In the following, we assume that $M$ (resp. $Z$) is usually the maximal ideal cycle $M_E$ (resp. fundamental cycle $Z_E$) on $(\tilde{X}, \tilde{E})$.

Lemma 2.1. ([6]) We have the following : coeff$_{\tilde{E}_i}$ $M = 2, \tilde{M}^2 = -2, Z \cdot E_i = 0$ and

$$
M \cdot E_i = \begin{cases} 
-1 & \text{if } i = 1 \\
0 & \text{if } i \neq 1 
\end{cases}
$$

Let $\tilde{E}_i, \cdots, \tilde{E}_n$ be all irreducible curves intersecting $\tilde{E}$, and $d_j = $ coeff$_{\tilde{E}_i}$ $\tilde{Z}$ for any $j$. By changing indices suitably we may suppose that $a_1, \cdots, a_s$ are odd and $a_{s+1}, \cdots, a_n$ are even. Since $\tilde{E}_i$ is an $(f)$-odd curve, $E_{i+1}, \cdots, E_n$ are irreducible from the rule in [5]. Hence it is obvious that $E - E_i$ decomposes into $m$ connected components. Let $F_j$ be a connected component containing $E_i$. We call $\{F_{i+1}, \cdots, F_k\}$ (resp. $\{F_{i+1}, \cdots, F_k\}$) odd (resp. even) connected components in $E$. Since $-1 = Z \cdot \tilde{E}_i = \tilde{E}_i^2 + \sum a_j$ and $\tilde{E}_i^2 = 2b_i^2$, there exists at least one odd connected component (i.e., $k \geq 1$). Let $E(1)$ be the union of all even
connected components and $E_i$, and let $E(2)$ be the union of all odd connected components. We call $E(1)$ (resp. $E(2)$) the even (resp. odd) part of $E$. Since $Z^2 = -1$ and Lemma 2.1, there is only one irreducible exceptional curve $E_p (p \neq 1)$ such that $Z \cdot E_p = -1$. Then the intersection number of $Z$ and any other component except for $E_p$ is 0.

**Lemma 2.2.** (i) $\text{Coeff} \{Z = 1\}$ and the number of odd connected components is one (i.e., $k=1$ and $E(2) = F_i$).

(ii) $E_p$ is contained in the odd connected component $F_i$.

(iii) The restrictions of $M$ and $Z$ onto the even part $E(1)$ have the relation $M|_{E(1)} = 2Z|_{E(1)}$.

Proof. From Lemma 2.1, we have $1 \leq \text{Coeff} \{Z = 1\} \leq \text{Coeff} \{M = 2\}$. Assume $\text{Coeff} \{Z = 2\}$. Then $-1 = M \cdot E_i = 2E_i^2 + (M - 2E_i) \cdot E_i$ and $0 = Z \cdot E_i = 2E_i^2 + (Z \cdot 2E_i) \cdot E_i$. But this contradicts the inequality $(M - 2E_i) \cdot E_i \geq (Z - 2E_i) \cdot E_i$. Then $\text{Coeff} \{Z = 1\}$. Since $E_p \neq E_i$, we assume that $E_p$ is contained in a connected component $F_i$. Let $\bar{F_i}$ be the union of connected components except for $F_p$, so $E = E_i \cup \bar{F_i}$. Let consider decompositions $Z = Z_1 + Z_2$ and $M = M_1 + M_2$, where $M_1 = M|_{\bar{F_i}}$, $Z_1 = Z|_{\bar{F_i}}$, $M_2 = M|_{E_i \cup F_i}$ and $Z_2 = Z|_{E_i \cup F_i}$. We consider any irreducible curve $E_{\bar{i}}$ in $F_i$ with $i \neq j$. If $E_{\bar{i}}$ intersects $E_i$, then we have $Z \cdot E_{\bar{i}} = (Z - Z_1) \cdot E_{\bar{i}} = -Z_2 \cdot E_{\bar{i}} = -E_i \cdot E_{\bar{i}} = -1$ and we can see $M_1 \cdot E_{\bar{i}} = -2$ similarly. Further, if $E_{\bar{i}}$ doesn't intersect $E_i$, then $Z_1 \cdot E_{\bar{i}} = (Z - Z_2) \cdot E_{\bar{i}} = -Z_2 \cdot E_{\bar{i}} = 0$ and also $M_1 \cdot E_{\bar{i}} = 0$. Therefore, by Cramer's rule, we have $M_1 = 2Z_1$ on $\bar{F_i}$. Let $(E_{\bar{i}}|_{\bar{F_i}})_{1, \ldots, n}$ and $(E_{\bar{i}}|_{\bar{F_i}})_{1, \ldots, k}$ be as above. Since $E_i$ is an $(f)$-odd curve, any $E_{\bar{i}}$ is an $(f)$-even curve from the assumption about a covering resolution in (2.1). Then $\text{Coeff} \{E_i \} = \text{Coeff} \{Z = a_j \}$ and this is an odd number for $i = 1, \ldots, k$.

Now we consider the relation between $M$ and $Z$ on $E(2)$. By changing the order of blowing-ups to get $E$, we may assume that $E(2) = \bigcup_{i=1}^{n} E_i$ and $E_2$ is only one irreducible curve in $E(2)$ which intersects $E_i$. We prove that $E_p$ above is equal to $E_2$.

**Lemma 2.3.** (i) $M|_{E(2)} = Z|_{E(2)}$.

(ii) $Z \cdot E_2 = -1 (\text{i.e., } E_p = E_2)$, $E_i \cdot E_2 = 1$ and $\text{Coeff} \{M = \text{Coeff} \{Z = 1\}$. 

Proof. Let $D = M - Z$. Then $D > 0$ from Lemma 2.1 and $-2 = M^2 = Z^2 + 2Z \cdot D + D^2$. Since $Z^2 = -1$ and $D^2 < 0$, we have $Z \cdot D = 0$ and $D^2 = -1$. Since $Z \cdot E_i = -1$, we have

- $(2.1)$ $E_p \not\subseteq \text{Supp}(D)$.

Then $\text{Coeff} \{M = \text{Coeff} \{Z = 1\}$. If the inverse image $\pi^{-1}(\pi(E_p))$ is a union of two disjoint irreducible curves $E_p \not\subseteq E_p$, then $Z \cdot E_p = Z \cdot E_p = -1$ and so $Z^2 \leq -2$. This contradicts $Z^2 = -1$, then $\pi^{-1}(\pi(E_p)) = E_p$. Also the configuration of $\pi(E(2))$ is a tree, then we can see that the support of $E - E_p$ decomposes into at most two disjoint connected components $E[1]$ and $E[2]$, where we assume $E_1 \subseteq E[1]$. Then $E = E[1] \cup E_p \cup E[2]$. It is possible that $E[2]$ is empty, here, by suitably exchanging indices, we may assume that $\bigcup_{i=1}^{n} E_i \subseteq E[1]$ and $\bigcup_{i=1}^{n} E_i \subseteq E[2]$. From (2.1) we have a decomposition $D^1 = D_1^1 + D_2^1$ such that $\text{Supp}(D_i) \subseteq E[1]$ for $i = 1, 2$. Since $-1 = D^1 = D_1^1 + D_2^1$ and $D_i \cdot E_1 = D_1 \cdot E_1 = (M - Z) \cdot E_1 = -1$ from Lemma 2.1, we have $D_1^1 = -1$ and $D_2^1 = 0$. Therefore $D_2 = 0$, so we have

- $(2.2)$ $\text{Supp}(D) \subseteq E[1]$ and $M = Z$ on $E_p \cup E[2]$.

Since $D \cdot E_p = (M - Z) \cdot E_p = -1$ and (2.1), there is only one irreducible curve $E_{p-1}$ in $E[1]$ which intersects $E_p$. Then we have

- $(2.3)$ $\text{Coeff} \{E_{p-1} \} = D_1 = 1$.

From now on we prove $p > 2$. We assume $p > 2$, so we have $D \cdot E_{p-1} = (M - Z) \cdot E_{p-1} = 0$. Further $D \cdot E_i = (M - Z) \cdot E_i = -1$ by Lemma 2.1. Hence we get the following:

- $(2.4)$ $D \cdot E_i = -1$ if $i = 1, \ldots, p-1$.
- $D \cdot (\text{any other curve in } E(1)) = 0$.

From (2.2) and Lemma 2.2 (i) and $E(1) \subseteq \text{Supp}(D)$, we have

- $(2.5)$ $\text{Supp}(D) = E[1]$.

In fact, if $\text{Supp}(D) \not\subseteq E[1]$, there is $E_i \subseteq E[1]$ satisfying $E_i \not\subseteq \text{Supp}(D)$ and $2 \leq j \leq p - 1$. Then there exists a curve $E_j$ with $D \cdot E_j > 0$ among such curves. But this contradicts (2.4), so (2.5) holds. Now let $\tilde{Z}$ be the fundamental cycle on $E[1]$, so $D \geq \tilde{Z}$ from (2.4).
We have \(1 = -D' \geq -Z' \geq 1\) and \(Z \geq Z\) from the definition of \(Z\). From a lemma in [2] (p.426) we also have

\[(2.6) \quad D = Z\text{ and } Z \geq D.\]

Since \(\text{coeff}_E Z = 1\), it is easy to check that \(Z|_{\mathcal{E}(1)} = \hat{Z}|_{\mathcal{E}(1)}\). We put \(D' = Z - D\). Then we have \(\text{Supp}(D') \subset \mathcal{E}(2) = \bigcup_{E_j} E_j\). Further we have \(D' \cdot E_p = (Z - \hat{Z}) \cdot E_p = -Z \cdot E_p - (M - Z) \cdot E_p = 0\) from \(p > 2\). Since \(Z \cdot E_p = 1\) by (2.3), this implies \(D' \cdot E_p = (Z - \hat{Z}) \cdot E_p = -2\).

Hence we have the following:

\[(2.7) \quad \begin{cases} D' \cdot E_p = 0 \text{ if } 2 \leq j \leq p - 1 \text{ or } p < j \\ D' \cdot E_p = -2. \end{cases}\]

Since \(M = Z + D = 2D + D'\) and \(M^2 = -2\) and \(D^2 = -1\), we have \(4D' \cdot D' + D' \cdot D = 2\). We put \(D' = \sum d_j E_j\), so \(D' = -2d_p\) from (2.7) and \(D^2 = d_p\) from (2.3) and (2.4). Hence \(d_p = 1\). From \(E_p \subset \text{Supp}(\hat{Z}) = \mathcal{E}(1)\) and \(Z = \hat{Z} + D'\), we have

\[(2.8) \quad \text{coeff}_{E_p} Z = 1.\]

Therefore we can easily see that \(Z = Z|_{\mathcal{E}(1)} + Z|_{\mathcal{E}(2)} = E_p\) and \(Z|_{\mathcal{E}(1)} = \hat{Z} = D\). Then \(M = 2Z|_{\mathcal{E}(1)} + Z|_{\mathcal{E}(2)} + E_p\) and \(\text{coeff}_{E_p} M = 2 \cdot \text{coeff}_{E_p} Z\) from \(E_p \subset \mathcal{E}(1)\). On the other hand, we have \(\text{coeff}_{E_p} Z = 1\) odd, this yields a contradiction. Then we have \(p = 2\). Hence \(M|_{\mathcal{E}(1)} = Z|_{\mathcal{E}(1)}\) and \(E(1) = \mathcal{E}(1)\) and \(E(2) = \mathcal{E}(2) \cup E_p\).

Since \(M = Z + Z|_{\mathcal{E}(1)}\), we have \(-1 = M \cdot E_i = (Z|_{\mathcal{E}(1)}) \cdot E_i = (Z - (\text{coeff}_{E_p} Z) E_p) \cdot E_i = -\text{coeff}_{E_p} Z\). Then \(\text{coeff}_{E_p} M = 1\). Q.E.D.

**Theorem 2.4.** Assume \((X, o)\) is a normal double point of type \((1,2)\). Let \((\hat{X}, A)\) be the minimal resolution or the minimal good resolution of \((X, o)\). Let \(M_A\) (resp. \(Z_A\)) be the maximal ideal (resp. fundamental) cycle on \(A\).

(i) The exceptional set \(A\) has a decomposition \(A = A(1) \cup A(2)\) with following two properties:

(a) \(A(1)\) and \(A(2)\) are non-empty connected 1-dimensional subvarieties of \(A\) without common irreducible curves and satisfy \(A(1) \cdot A(2) = 1\).

(b) Let \(M|_{A(i)}\) and \(Z|_{A(i)}\) be the restrictions of \(M_A\) and \(Z_A\) onto \(A(i)\) for \(i = 1, 2\). Then \(M|_{A(1)} = Z|_{A(1)}\) and \(M|_{A(2)} = Z|_{A(2)}\).

(ii) Let \(A_1 \subset A(1)\) and \(A_2 \subset A(2)\) be two irreducible curves with \(A_1 \cdot A_2 = 1\), where \(A_i\) exists uniquely for \(i = 1, 2\) from (i). Then \(M \cdot A_i = -1\), \(\text{coeff}_{A_i} M = 2\), \(Z \cdot A_i = -1\) and \(\text{coeff}_{A_i} Z = 1\).

Moreover, if \((X, o)\) is of type \((1,1)\), then \(M_A = Z_A\) on \((\hat{X}, A)\).

**Proof.** Let \((\hat{X}, E)\) be a good resolution space as in (1.1). By contracting \((-1)\)-curves suitably, we get the minimal good resolution space \((\hat{X}, A)\). Let \(\varphi : (\hat{X}, E) \rightarrow (\hat{X}, A)\) be a such contraction map. If \(E_i\) is an irreducible curve in \(E\) which is not contracted to a point, then we have \(\text{coeff}_{E_i} Z_A = \text{coeff}_{E_i} Z_A\) and \(\text{coeff}_{E_i} M_A = \text{coeff}_{E_i} M_A\). Therefore, if every curve of \(E(1)\) is contracted by \(\varphi\), then \(M_A = Z_A\) and \(M_A^2 = -1\). It contradicts the assumption of type \((1,2)\). Then the even part \(E(1)\) is not contracted to a point under \(\varphi\). Further the odd part \(E(2)\) is not contracted through \(\varphi\). Because if \(\varphi(E(2))\) is a point, then \(M_A = 2Z_A\) on the minimal good resolution, so \(M_A^2 = -4\). This yields a contradiction. Moreover we can easily see that if \(E_1\) (resp. \(E_2\)) is contracted to a point through \(\varphi\), then \(\text{coeff}_{E_i} Z_A\) (resp. \(\text{coeff}_{E_i} Z_E\)) is larger than one. It contradicts Lemma 2.2 and 2.3. Therefore \(E_1\) and \(E_2\) are not contracted through \(\varphi\). Then if we put \(A(1) = \varphi(E(1))\), \(A(2) = \varphi(E(2))\), \(A_1 = \varphi(E_1)\), and \(A_2 = \varphi(E_2)\), then these subvarieties satisfy the conditions of Theorem 2.4. For the case of the minimal resolution we can prove it similarly.

Assume \((X, o)\) is of type \((1,1)\). If \((\hat{X}, A)\) is the minimal resolution, then \(M_A = Z_A\). Then we consider the case of the minimal good resolution. Let \((\check{X}, E)\) be a good resolution of \((X, o)\) as in (1.1). From Lemma 2.2 and 2.3, it is easy to check that there is a contraction map \(\gamma\) of all curves in \(E(1)\) such that \(\gamma\) doesn't contract any curve in \(E(2)\). Since \(E(2)\) is simple normal crossing on \(\gamma(\check{X})\) and we have \(M_E = Z_E\) on the good resolution space \((\check{X}, E)\), the equality also holds on the minimal good resolution. Q.E.D.

We call \(A(1)\) (resp. \(A(2)\)) the even (resp. odd) part of \(A\).

**Remark 2.5.** U. Karras [7] introduced the notion of Kodaira singularities in terms of pencils of curves. Let \(\pi : (\check{X}, A) \rightarrow (X, o)\) be a minimal good resolution of a normal complex surface singularity. In [8], he also proved that \((X, o)\) is a Kodaira singularity if and only if the maximal ideal cycle \(M_A\) and the fundamental cycle \(Z_A\) on \(A\) are equal and the w.d. graph is a Kodaira graph (i.e., \(\text{coeff}_{A} Z_A = 1\) for every \(A_i\) with \(Z_A \cdot A_i < 0\)). In [9], the author proved
that if \((X, o) = (x^n = f(x, y))\) and \(ord(f)\) is divided by \(n\), then \((X, o)\) is a Kodaira singularity (also see [10]).

Let \((X, o) = (\tilde{z} = f(x, y))\) be a normal double point. Then if \(ord(f)\) is even, \((X, o)\) is a Kodaira singularity. Hence we suppose \(ord(f)\) is odd. If \((X, o)\) is of type \((2,2)\) or \((1,2)\), then it is not a Kodaira singularity. Because if \((\tilde{X}, E)\) is a covering resolution, then we have \(Z \cdot E_i = -1\) and \(\text{coeff}_E Z = 2\) for \((2,2)\) and \(Z_E < M_E\) for \((1,2)\). These relations are kept onto \(A\) through a contraction \(\tilde{X} \rightarrow \tilde{X}\), so we have the above. Next suppose that \((X, o)\) is of type \((1,1)\). It is obvious that the w.d. graph on the minimal good resolution is a Kodaira graph from \(Z^2 = -1\). Further we proved in Theorem 2.4 that \(M = Z\) on the minimal good resolution. Then \((X, o)\) is a Kodaira singularity from Karras's result.

**Example 2.6.** Let \((X, o)\) be a normal double point \([z^2 = y(y^2 + x^3)(x^3 + y^6)] \subset (C, 0)\). The w.d. graph of the minimal good resolution is given as follows:

```
      A_6
     /   \
    /     \
   A_1   A_2
  /   \
-1   -6

[1]
```

Then we have \(A(1) = A_1 \cup A_6\) and \(A(2) = \bigcup A_i\). From Theorem 2.4 we have \(M_A = 2A_6 + 2A_1 + A_2 + 4A_3 + 2A_4 + A_6\) and \(Z_A = A_6 + A_1 + A_2 + 4A_3 + 2A_4 + A_6\). Hence \((X, o)\) is of type \((1,2)\) and \(M_A \cdot A_1 = Z_A \cdot A_2 = -1\).

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