A formula of the fundamental genus for hypersurface singularities of Brieskorn type.

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Received November 30, 1996

Key Words : Singularity, Fundamental genus

Summary : In [6] the author proved a formula for the fundamental genus of hypersurface singularities defined by Brieskorn type with three variables. In present paper we prove a more generalized formula for such singularities.

0. Introduction

Let \( \pi : (\tilde{X}, A) \to (X, z) \) be a resolution of a normal surface singularity, where \( \pi^{-1}(z) = A = \bigcup_{i=1}^{n} A_i \) is the irreducible decomposition of the exceptional set \( A \). For a cycle \( D = \sum_{i=1}^{n} d_i A_i \ (d_i \in \mathbb{Z}) \) on \( A \), \( \chi(D) \) is defined by \( \chi(D) = \text{dim}_c H^0(\tilde{X}, \mathcal{O}_D) - \text{dim}_c H^1(\tilde{X}, \mathcal{O}_D) \), where \( \mathcal{O}_D = \mathcal{O}_{\tilde{X}} / \mathcal{O}(-D) \). Then \( \chi(D) = -\frac{1}{2}(D^2 + DK_{\tilde{X}}) \), where \( K_{\tilde{X}} \) is the canonical sheaf (or divisor) on \( \tilde{X} \). For any irreducible component \( A_i \), we have \( K_{\tilde{X}} A_i = -A_i^2 + 2g(A_i) - 2 + 2\delta(A_i) \) (adjunction formula), where \( g(A_i) \) is the genus of the non-singular model of \( A_i \) and \( \delta(A_i) \) is the degree of the conductor of \( A_i \) (cf. [2]). The arithmetic genus of \( D \geq 0 \) is defined by \( p_a(D) = 1 - \chi(D) \). The fundamental cycle on \( A \) is defined as follows (cf. [1]):

\[ Z = \min \{ D = \sum_{i=0} a_i A_i | DA_i \leq 0 \text{ for any } i \text{ and } a_i > 0 \} \]

The following number for surface singularities is defined by (cf. [7])
$p_f = p_f(X, z) = p_a(Z)$ (fundamental genus),
and this is a topological invariant and independent of the choice of a resolution
of $(X, z)$. We call $p_f(X, z)$ the fundamental genus of $(X, z)$.

In this paper we consider the fundamental genus of hypersurface singularities
of Brieskorn type with degree $(a_0, a_1, a_2)$ (i.e., $(X, z) = \{x_0^{a_0} + x_1^{a_1} + x_2^{a_2} = 0\} \subseteq \mathbb{C}^3$). For them, the author has proved in [6] that $p_f(X, z) = \frac{1}{2} \{(a_0 - 1)(a_1 - 1) - (a_0, a_1) + 1\}$ under the condition: $a_2 \geq l.c.m.(a_0, a_1)$.

(cf. Theorem 4.3. [6]). We improve this formula in the following.

Notations. For integers $a_1, a_2, \ldots, a_n$ ($n \geq 2$), we put

$$[a_1, a_2, \ldots, a_n] := a_1 - \frac{1}{a_2 - \frac{1}{\ddots - \frac{1}{a_n}}}$$

(continued fraction).

For real number $a$, we put $[a] := \max \{n \in \mathbb{Z} | n \leq a\}$ (Gauss symbol) and

$\{a\} = \min \{n \in \mathbb{Z} | n \geq a\}$. Further, for positive integers $a_1, \ldots, a_n$, we put

$$(a_1, \ldots, a_n) := g.c.m.(a_1, \ldots, a_n).$$

1. Preliminaries First we describe a resolution of a singularity defined
by $x_0^{a_0} + x_1^{a_1} + x_2^{a_2}$. For integers $a_0, a_1$ and $a_2$, we denote the following integers:

$$d_6 = (a_0, a_1, a_2), d_3 = \frac{(a_1, a_2)}{d_6}, d_4 = \frac{(a_0, a_2)}{d_6},$$

$$d_5 = \frac{(a_0, a_1)}{d_6}, d_0 = \frac{a_0}{d_4 d_5 d_6}, d_1 = \frac{a_1}{d_3 d_5 d_6}, d_2 = \frac{a_2}{d_3 d_4 d_6}.$$ Further, let $e_i$ ($i = 0, 1, 2$) be an integer defined by $e_i; 1 + 1 \equiv 0(d_i)$ and $0 < e_i < d_i$. If $\pi : (\tilde{X}, A) \rightarrow (X, z)$ is a resolution of a normal surface singularity with
$\mathbb{C}^*$-action. Then the configuration of the weighted dual graph is a star-shaped
graph. Especially, the weighted dual graph associated to $x_0^{a_0} + x_1^{a_1} + x_2^{a_2}$ is given by following forms:

[Diagram of a star-shaped graph]
where $(b_{i1}, \ldots, b_{ir})$ means cyclic branch determined by the cyclic quotient singularity $C_{d_{i1}}$, and $[b_{i1}, \ldots, b_{ir}] = \frac{d_i}{e_i}$ ($i = 0, 1, 2$). Their all components are $\mathbb{P}^1$. $A_0$ is a curve of genus $g$, which is called the central curve.

In this situation we denote $\mathbb{Q}$-coefficient divisor $D$ and $\mathbb{Z}$-coefficient divisor $[kD]$ on $A_0$ as follows:

$$D := D_0 - \sum_{i=0}^{2} \sum_{j=1}^{d_{i+3}} \frac{e_i}{d_i} P_{ij} \quad \text{and} \quad [kD] := kD_0 - \sum_{i=0}^{2} \sum_{j=1}^{d_{i+3}} \left\{ \frac{ke_i}{d_i} \right\} P_{ij},$$

where $k$ is a non-negative integer and $D_0$ is a divisor on $A_0$ such that $\mathcal{O}_{A_0}(D_0)$ is the restriction to $A_0$ of the conormal sheaf of $A_0$ in $\bar{X}$. Let’s $m$ be an integer defined by $\min\{k \in \mathbb{Z} | \deg[kD] \geq 0\}$.

2. Main Results

Lemma 1. If $a_0 \leq a_1 \leq a_2$, then $m = \min\{d_0d_1d_2, d_0d_1d_3\}$.

Proof. First we prove the following:

\begin{equation}
(2.1) \quad m = l_2 \text{ if } d_2 \geq d_3.
\end{equation}

From the definition of $[kD]$, we have

$$\deg[kD] = d_6 \left\{ \frac{k}{d_0d_1d_2} - \sum_{i=0}^{2} d_{3+i}(\left\{ \frac{ke_i}{d_i} - \frac{ke_i}{d_i} \right\}) \right\}.$$ 

Therefore we can easily see the following:

$$\deg[l_2D] = d_6 \left\{ \frac{l_2}{d_0d_1d_2} - d_5 \left\{ \frac{2e_2}{d_2} \right\} + \frac{d_3l_2e_2}{d_2} \right\}$$

$$= d_5d_6 \left\{ \frac{l_2}{d_2} - \left\{ \frac{2e_2}{d_2} \right\} + \frac{2e_2}{d_2} \right\} = 0$$

from the relation: $l_2e_2 + 1 \equiv 0(d_2)$. Hence we may only prove that

\begin{equation}
(2.2) \quad \deg[kD] < 0 \text{ for any } k \text{ with } 0 < k < l_2.
\end{equation}

For such integer $k$, if $d_2 \nmid k$, then

$$\frac{1}{d_6} \deg[kD] \leq \frac{k}{d_0d_1d_2} - \sum_{i=0}^{2} d_{3+i}(\left\{ \frac{ke_i}{d_i} - \frac{ke_i}{d_i} \right\}) < 0.$$ 

We prove (2.2) for the case $d_2|k$. Let $k = d_2t$. Then

$$\frac{1}{d_6} \deg[kD] = \frac{t}{d_0d_1} - \sum_{i=0}^{1} d_{3+i}(\left\{ \frac{d_2e_it}{d_i} - \frac{d_2e_it}{d_i} \right\}).$$

If $d_0|t$ and $d_1|t$, then $d_0d_1d_2(= l_2)|t$. It contradicts the inequality $k < l_2$. Assume that $d_0 \nmid t$. Then

$$\frac{1}{d_6} \deg[kD] \leq \frac{t}{d_0d_1} - d_3(\left\{ \frac{d_2e_0t}{d_0} - \frac{d_2e_0t}{d_0} \right\})$$

$$\leq \frac{t}{d_0d_1} - d_3 = \frac{d_2t - l_0}{d_0d_1d_2} = \frac{k - l_0}{d_0d_1d_2}.$$ 

From the assumption $a_0 \leq a_1$, we have $l_2 \leq l_0$. Then $k - l_0 < 0$, so $\deg[kD] < 0$.

Next assume that $d_1 \nmid t$. Then

\[ \text{...} \]
\[
\frac{1}{d_6} \deg[kD] \leq \frac{t}{d_0 d_1} - d_4 \left( \frac{d_2 \varepsilon_1 t}{d_1} - \frac{d_3 \varepsilon_1 t}{d_1} \right)
\leq \frac{t}{d_0 d_1} - \frac{d_4}{d_0} = \frac{d_4}{d_0 d_1} \frac{d_4}{d_2} = \frac{k - l_1}{d_0 d_1 d_2}.
\]

From the assumption \(a_1 \leq a_2\), we have \(l_2 \leq l_1\). Then \(k - l_1 < 0\), so \(\deg[kD] < 0\).

Second we prove the following:

\[
(2.3) \quad m = d_0 d_1 d_2 \text{ if } d_2 < d_5.
\]

Since \(\deg[d_0 d_1 d_2 D] = d_6 > 0\), we may only prove that \(\deg[kD] < 0\) for any \(k\) with \(0 < k < d_0 d_1 d_2\). For an integer \(k\) as above, \(d_i\) doesn't divide \(k\) for some \(i \in \{0, 1, 2\}\). Assume that \(d_0 \nmid k\). Then

\[
\frac{1}{d_6} \deg[kD] = \frac{k}{d_0 d_1 d_2} - \sum_{i=0}^{2} d_{3+i}(\{\frac{ke_i}{d_i}\} - \frac{ke_i}{d_i})
\leq \frac{k}{d_0 d_1 d_2} - \frac{d_3}{d_0} = \frac{k - l_0}{d_0 d_1 d_2}.
\]

Since \(d_2 < d_5\) and \(k < d_0 d_1 d_2 < l_2 \leq l_0\), we have \(\deg[kD] < 0\). Q.E.D.

**Theorem 2.** For hypersurface singularity \((X, z) = \{x_0^{a_0} + x_1^{a_1} + x_2^{a_2} = 0\} \subset C^3\), we have following formulae.

1. If \(m = l_2\), then \(p_f(X, z) = \frac{1}{2} \left( (a_0 - 1)(a_1 - 1) - (2\frac{d_2}{d_0} + 1)(a_0, a_1) + 1 \right)\).
2. If \(m = d_0 d_1 d_2\), then \(p_f(X, z) = \frac{d_4}{2} \left( a_0 l_0 - m - l_0 - l_1 - l_2 + 1 \right) + 1\).

Proof. By the formula of \(p_f(X, z)\) due to Masataka Tomari (Theorem 3.1, [6]) and Riemann-Roch Theorem,

\[
p_f(X, z) = \sum_{k=0}^{m-1} \dim_C H^0(A_0, \mathcal{O}_C(KA_0 - [kD]))
= g + \sum_{k=1}^{m-1} \{\deg(KA_0 - [kD]) - g + 1\}
= g + \sum_{k=1}^{m-1} \{g - 1 - \deg[kD]\} = m(g - 1) + 1 - \sum_{k=1}^{m-1} \deg[kD].
\]

We have

\[
\sum_{k=1}^{m-1} \deg[kD] = \frac{d_4}{d_0 d_1 d_2} \cdot \frac{m(m - 1)}{2} - d_6 \sum_{k=1}^{m-1} \sum_{i=0}^{2} d_{3+i}(\{\frac{ek_i}{d_i}\} - \frac{ek_i}{d_i}).
\]

Now we prove (1), so assume \(m = l_2(= d_0 d_1 d_5)\). Then

\[
\sum_{k=1}^{l_2-1} \left( \{\frac{ek_0}{d_0}\} - \frac{ek_0}{d_0} \right) = \frac{d_4 d_5 (d_0 - 1)}{2},
\]

\[
\sum_{k=1}^{l_2-1} \left( \{\frac{ek_1}{d_1}\} - \frac{ek_1}{d_1} \right) = \frac{d_6 d_5 (d_1 - 1)}{2},
\]

\[
\sum_{k=1}^{l_2-1} \left( \{\frac{ek_2}{d_2}\} - \frac{ek_2}{d_2} \right) = \frac{(l_2 - 1)(d_2 + 1) - l_2}{2d_2}.
\]

Therefore we have the following:

\[
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\]
\[
\frac{d_0d_1d_2}{d_6} \sum_{k=0}^{l_2-1} \deg[kD] = \sum_{k=1}^{l_2-1} k - d_0d_1d_2 \sum_{i=0}^{2} d_3 + i \sum_{k=1}^{l_2-1} \left( \left\{ \frac{e_i}{d_i} \right\} - \frac{e_i}{d_i} \right)
\]
\[
= \frac{l_2(l_2-1)}{2} - d_0d_1d_2d_5 \left\{ \frac{d_1d_3(d_0-1)}{2} + d_0d_4(d_1-1) \right\}
\]
\[
+ \left( \frac{l_2-1}{2d_2} + \frac{l_2}{d_2} \right)
\]
\[
= \frac{l_2}{2} \left\{ \frac{l_2-1}{2} - \frac{(l_2-1)(d_2+1)}{2} - \frac{l_0(d_0+1)}{2} - \frac{l_1(d_1-1)}{2} + d_2 \left[ \frac{l_2}{d_2} \right] \right\}
\]
\[
= \frac{l_2}{2} \left\{ -l_2d_2 + d_2 - l_0d_0 + l_0 - l_1d_1 + l_1 + 2d_2 \left[ \frac{l_2}{d_2} \right] \right\}.
\]

From the computations above, we obtain the following.
\[
p_f(X, z) = \frac{l_2(d_3d_4d_5d_6^2 - d_0d_3 - d_6d_4 - d_6d_5)}{2} + 1
\]
\[
- \frac{d_5}{2d_2} \left\{ -l_2d_2 + d_2 - l_0d_0 + l_0 - l_0d_1 + l_1 + 2d_2 \left[ \frac{l_2}{d_2} \right] \right\}
\]
\[
= 1 + \frac{1}{2} \left\{ a_0a_1 - a_0 - a_1 - (a_0, a_1) - 2d_5d_6 \left[ \frac{l_2}{d_2} \right] \right\}
\]
\[
= \frac{1}{2} \left\{ (a_0-1)(a_1-1) - \frac{(l_2^2)}{d_2} + 1 \right\}.
\]

Now we prove (2), so \( m = d_0d_1d_2 \). We have the following equality (see [6], Lemma 4.2):
\[
\sum_{k=1}^{d_0d_1d_2-1} \left( \left\{ \frac{ke_0}{d_0} \right\} - \frac{ke_0}{d_0} \right) = \frac{d_1d_3(d_0-1)}{2}.
\]

Then
\[
p_f(X, z) = m(g-1) + 1 - \frac{d_6}{2} \left( m - \frac{1}{2} \right) + \frac{d_6}{2} \sum_{i=0}^{2} d_3 + i \sum_{k=1}^{m-1} \left( \left\{ \frac{e_i}{d_i} \right\} - \frac{e_i}{d_i} \right)
\]
\[
= m(g-1) - \frac{d_6}{2} + \frac{d_6}{2} \left\{ d_1d_2d_3(d_0-1) + d_0d_2d_4(d_1-1) + d_0d_1d_5(d_2-1) \right\}
\]
\[
= m(g-1) - \frac{d_6}{2} + \frac{d_6}{2} + \frac{md_6}{2} \left( d_3 + d_4 + d_3 \right) - \frac{d_6}{2} \left( l_0 + l_1 + l_2 \right)
\]
\[
= \frac{md_6}{2} \left( d_3d_4d_5d_6 - 1 \right) + \frac{d_6}{2} \left( l_0 + l_1 + l_2 \right) + \frac{d_6}{2} + 1
\]
\[
= \frac{d_6}{2} \left\{ a_0l_0 - m - l_0 - l_1 - l_2 + 1 \right\} + 1. \text{Q.D.E.}
\]

**Corollary 3** (cf. Theorem 4.3 in [6]). If \( a_2 \geq l.c.m.(a_0, a_1) \), then
\[
p_f(X, z) = \frac{1}{2} \left( (a_0-1)(a_1-1) - (a_0, a_1) + 1 \right).
\]

**Proof.** By the assumption, we have \( m = l_2 \) and \( \left[ \frac{l_2}{d_2} \right] = 0 \). From (1) of Theorem 2 we obtain the result. Q.E.D.
Example 4. Let \((X, z) = \{z_0^6 + z_1^{10} + z_2^{d_2}\}\). If we vary \(d_2\), we have the following table of \(p_f(X, z)\):

\[
\begin{array}{cccccccc}
  \quad & a_2 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\
\end{array}
\]

In the table above, we have \(m = d_0d_1d_2 = 1\) for the case of \(a_2 = 15\).

REFERENCES


