A Torsion Theory for the Category of Finite Groups

Koichiro OHTAKE
Department of Mathematics, Faculty of Education, Gunma University

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Abstract

In this paper we define a torsion theory in the category of finite groups. We prove that there exists one to one correspondence between idempotent radicals (which are special subfunctors of the identity functor of the category of finite groups) and torsion theories. We also prove that a class of finite groups is a torsion free class if and only if it is closed under subgroups and group extensions. A typical example of torsion free classes is the class of solvable groups.

0 Introduction

A torsion theory was defined by S.E. Dickson[1] for abelian categories. Torsion theories are very useful to generalize Morita equivalence, Morita duality and localizations of rings. They are considered in full subcategories of module categories over rings with identity. When we consider abelian categories, they are always full subcategories of module categories. The author has been considered localization and colocalization with respect to torsion theories in abelian categories. But abelian categories were always full subcategories of modules categories as was mentioned above. The author wanted to apply the idea of localization and colocalization to other categories which are not full subcategories of module categories.

In this paper we define a torsion theory in the category of finite groups by means of a functor called an idempotent radical. A torsion theory is a pair of classes of finite groups. Suppose $(\mathcal{T}, \mathcal{F})$ is a torsion theory. Then $\mathcal{T}$ is called a torsion class and $\mathcal{F}$ is called a torsion free class. We will show that a class of finite groups is a torsion free class if and only if it is closed under subgroups and group extensions. The dual assertion does not hold. There exists a class of finite groups which is closed under epimorphic images and group extensions, but it is not a torsion class of any torsion theory. Therefore it is natural to think colocalization with respect to a torsion theory rather to think localization. But we will not discuss colocalization in this paper.

1 Torsion theories in the category of finite groups

$\mathcal{G}$ denotes the category of finite groups, namely the objects of $\mathcal{G}$ are finite groups and morphisms are homomorphisms. In $\mathcal{G}$ an epimorphism is a surjective homomorphism and a monomorphism is an injective homomorphism. $1_{\mathcal{G}}$ denotes the identity functor of $\mathcal{G}$. Let t be a subfunctor of $1_{\mathcal{G}}$, that is, t is an endofunctor of $\mathcal{G}$ and there is a natural transformation $\Phi : t \to 1_{\mathcal{G}}$ such that $\Phi_G : t(G) \to G$ is a monomorphism for any $G \in \mathcal{G}$. In this case t is also called a preradical. In this paper we assume that if t is a preradical then the natural map $\Phi_G : t(G) \to G$ is an inclusion map for each G. A preradical t is called idempotent if $t^2 = t$, that is, $t(t(G)) = t(G)$ for all G. Moreover we call a preradical t a radical if $t(G) \lhd G$ and $t(G/t(G)) = 1$ for all G. If $f : X \to Y$ is a morphism in $\mathcal{G}$ such that $f(x) = 1$ for all $x \in X$ then we call f null and denote it by 0.

Throughout this section let t be an idempotent radical in $\mathcal{G}$. Let $\mathcal{T} = \{\, x \in \mathcal{G} \mid t(X) = X \,\}$ and $\mathcal{F} = \{\, x \in \mathcal{G} \mid t(X) = 1 \,\}$. We call $(\mathcal{T}, \mathcal{F})$ a torsion theory associated with t. And we call $\mathcal{T}$ the torsion class and $\mathcal{F}$ the torsion free class of the torsion theory.

Theorem 1.1 *$\mathcal{T}$ and $\mathcal{F}$ have the following properties.*
(1) If $X \in \mathcal{T}$ and $Y \in \mathcal{F}$ then $Hom(X, Y) = 0$, that is, $Hom(X, Y)$ consists of only a null morphism.
(2) $\mathcal{T}$ is closed under epimorphic images.
(3) $\mathcal{F}$ is closed under subgroups.
(4) $\mathcal{T}$ and $\mathcal{F}$ are closed under group extensions.

Proof *(1) Let $f : X \to Y$ be a morphism in $\mathcal{G}$. The we have the following commutative diagram.*

$$\begin{array}{ccc} t(X) & \xrightarrow{t(f)} & t(Y) \\ \| & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Since $t(Y) = 1$, $t(f) = 0$. This implies $f = 0$.
(2) Let $X \to Y$ be an epimorphism with $X \in \mathcal{T}$. Then we have the following commutative diagram with the exact bottom row.

$$\begin{array}{ccccc} t(X) & \longrightarrow & t(Y) & & \\ \| & & \downarrow & & \\ X & \longrightarrow & Y & \longrightarrow & 1 \end{array}$$

Then it is clear that $t(X) \to t(Y)$ is an epimorphism. This implies $t(Y) = Y \in \mathcal{T}$.
(3) This can be proved as the dual case of (2). So we omit the proof.
(4) We show that $\mathcal{T}$ is closed under group extensions. Let $1 \to X' \to X \to X'' \to 1$ be exact with $X', X'' \in \mathcal{T}$. We show $X \in \mathcal{T}$. We have the following commutative diagram with the exact bottom row.

$$\begin{array}{ccccccccc} & & t(X') & \longrightarrow & t(X) & \longrightarrow & t(X'') & & \\ & & \| & & \downarrow & & \| & & \\ 1 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 1 \end{array}$$

Since $X' \subset t(X)$, there exists an epimorphism $X'' \to X/t(X)$. But since $X'' \in \mathcal{T}$ and $X/t(X) \in \mathcal{F}$, it follows $Hom(X'', X/t(X)) = 0$. This implies $X/t(X) = 1$, that is, $X = t(X) \in \mathcal{T}$. We omit the proof of the case of $\mathcal{F}$ since it can be proved dually.

The following theorem implies that there exists one to one correspondence between torsion theories and idempotent radicals.

Theorem 1.2 *Suppose r is an idempotent radical with $\{ X \in \mathcal{G} \mid r(X) = X \} = \mathcal{T}$ and $\{ X \in \mathcal{G} \mid r(X) = 1 \} = \mathcal{F}$. Then $r = t$ holds.*

Proof *Consider the following diagram.*

$$\begin{array}{ccc} t(X) & & r(X) \\ & \searrow \quad \swarrow & \\ & X & \\ & \swarrow \quad \searrow & \\ X/r(X) & & X/t(X) \end{array}$$

Since $t(X) \in \mathcal{T}$ and $X/r(X) \in \mathcal{F}$, $Hom(t(X), X/r(X)) = 0$. Hence $t(X) \subset r(X)$. Similarly $r(X) \subset t(X)$ holds. Therefore $t(X) = r(X)$ holds. This implies $r = t$.

By the above theorem, if t is an idempotent radical then sometimes $\mathcal{T}_t$ and $\mathcal{F}_t$ denote the torsion class and the torsion free class of the torsion theory associated with t.

Let F be an endofunctor of $\mathcal{G}$. We say that F is left exact if $1 \to F(X') \to F(X) \to F(X'')$ is exact for an exact sequence $1 \to X' \to X \to X'' \to 1$. Similarly right exact can be defined. If F is left and right exact then we say that F is exact. Also we say that F is epi-preserving if $F(X) \to F(X'')$ is an epimorphism for an epimorphism $X \to X''$.

Proposition 1.3 *$\mathcal{T}$ is closed under normal subgroups if and only if t is left exact.*

Proof *Suppose that $\mathcal{T}$ is closed under normal subgroups. Let $1 \to X' \to X \to X'' \to 1$ be exact. Since t is a preradical, $1 \to X' \to X$ is exact. Hence it is enough to show $t(X') = Ker(t(X) \to t(X''))$ since we may assume that X' is a subgroup of X. We have the following commutative diagram with the exact bottom row.*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & t(X') & \longrightarrow & t(X) & \longrightarrow & t(X'') & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 1 \end{array}$$

Put $K = Ker(t(X) \to t(X''))$. It is enough to show that $K = t(X')$. Since $K \lhd t(X)$, $t(K) = K$ holds by assumption. Since $K \subset X'$, $t(K) \subset t(X')$. On the other hand, since $t(X') \subset K$, $t(t(X')) \subset t(K)$. Since t is idempotent, $t(X') \subset t(K)$. Hence $t(X') = t(K) = K$ holds. Therefore t is left exact. Conversely suppose that t is left exact. Let $X \in \mathcal{T}$ and X' a normal subgroup. Since $1 \to X' \to X/X' \to 1$ is exact, we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & t(X') & \longrightarrow & t(X) & \longrightarrow & t(X/X'') & & \\ & & \downarrow & & \| & & \downarrow & & \\ 1 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X/X' & \longrightarrow & 1 \end{array}$$

Then $t(X') = Ker(t(X) \to t(X/X')) = Ker(X \to X/t(X)) = X'$. Therefore $\mathcal{T}$ is closed under normal subgroups.

The next result is a dual case of the above result.

Proposition 1.4 *$\mathcal{F}$ is closed under epimorphic images if and only if t is epi-preserving.*

Proof *Suppose $\mathcal{F}$ is closed under epimorphic images. Let $f : G \to G''$ be an epimorphism. Put $N = f^{-1}(t(G''))$. Then $t(G) \lhd N \lhd G$. Since $f(t(G)) \subset t(G'')$ and $f(N) = t(G'')$, we obtain an epimorphism $N/t(G) \to t(G'')/f(t(G))$. Hence $t(G'')/f(t(G)) \in \mathcal{T} \cap \mathcal{F} = \{1\}$. This implies $t(G) \to t(G'')$ is an epimorphism, namely, t is epi-preserving. Conversely suppose that t is epi-preserving. Let $f : G \to G''$ be an epimorphism with $G \in \mathcal{F}$. Then by assumption $t(G) \to t(G'')$ is an epimorphism. But $t(G) = 1$. Hence $t(G'') = 1$. This implies $G'' \in \mathcal{F}$.*

Proposition 1.5 *The following assertions hold.*
(1) $\mathcal{T} = \{X \in \mathcal{G} \mid Hom(X, Y) = 0 \text{ for all } Y \in \mathcal{F}\}$.
(2) $\mathcal{F} = \{Y \in \mathcal{G} \mid Hom(X, Y) = 0 \text{ for all } X \in \mathcal{T}\}$.

Proof (1) *Put $\mathcal{T}' = \{X \in \mathcal{G} \mid Hom(X, Y) = 0 \text{ for all } Y \in \mathcal{F}\}$. Then by Theorem 1.1 $\mathcal{T} \subset \mathcal{T}'$. Conversely let $X \in \mathcal{T}'$. Then $Hom(X, X/t(X)) = 0$ since $X/t(X) \in \mathcal{F}$. This implies $X = t(X) \in \mathcal{T}$. Hence $\mathcal{T} = \mathcal{T}'$ holds. The proof of* (2) *is similar to* (1).

2 Torsion free classes

In this section we give a necessary and sufficient condition about a class of finite groups in order to be a torsion free class of some torsion theory. The dual assertion does not hold. We give an example of this. Also as a result of the argument, we will know that if t is an idempotent radical then $t(G)$ is a characteristic subgroup of G for all $G \in \mathcal{G}$.

Theorem 2.1 *Let $\mathcal{F}$ be a class of finite groups. Then $\mathcal{F}$ is a torsion free class of some torsion theory if and only if $\mathcal{F}$ is closed under subgroups and group extensions.*

Proof *"If" part has already been proved in Theorem 1.1. Suppose $\mathcal{F}$ is closed under subgroups and group extensions. Let $G \in \mathcal{G}$ and A, B normal subgroups of G such that G/A, $G/B \in \mathcal{F}$. Since AB/B is a subgroup of $G/B \in \mathcal{F}$, $AB/B \in \mathcal{F}$. Moreover $AB/B \simeq A/A \cap B \lhd G/A \cap B$. Hence we have an exact sequence $1 \to A/A\cap B \to G/A \cap B \to G/A \to 1$ with $A/A \cap B$, $G/A \in \mathcal{F}$. Thus $G/A \cap B \in \mathcal{F}$. Therefore there exists the smallest normal subgroup of G such that the group factored by it is in $\mathcal{F}$. Let us denote it by $t(G)$. We show that $t(G)$ char G (char denotes a characteristic subgroup). Let $\sigma \in Aut(G)$. Then $t(G)^\sigma \subset G^\sigma = G$ implies $G/t(G)^\sigma = G^\sigma/t(G)^\sigma \simeq G/t(G) \in \mathcal{F}$. Then by the definition of $t(G)$, $t(G) \subset t(G)^\sigma$. This means $t(G)$ char G. Next we show that t is idempotent. Suppose $t^2(G) \neq t(G)$. Then $1 \neq t(G)/t^2(G) \in \mathcal{F}$. On the other hand, $t^2(G)$ char $t(G)$ and $t(G)$ char G implies $t^2(G)$ char G. Moreover we have an exact sequence $1 \to t(G)/t^2(G) \to G/t^2(G) \to G/t(G) \to 1$ with $t(G)/t^2(G)$, $G/t(G) \in \mathcal{F}$. Therefore $G/t^2(G) \in \mathcal{F}$. Then by the definition of $t(G)$, $t(G) \subset t^2(G)$. This is a contradiction. Therefore t is idempotent. Here we show that $\mathcal{F} = \{Y \in \mathcal{G} \mid t(Y) = 1\}$ holds. If $t(Y) = 1$ then $Y/t(Y) = Y \in \mathcal{F}$, and if $Y \in \mathcal{F}$ then by the definition of $t(Y)$, $t(Y) = 1$. Suppose $X = t(X)$ and $X' \lhd X$. Then we show that $t(X/X') = X/X'$. If $t(X/X') \neq X/X'$ then $t(X/X') = K/X'$ for some $K \lhd X$ with $K \neq X$. Then $X/K \simeq (X/X')/t(X/X') \in \mathcal{F}$. This fact implies $t(X) \subset K$. This is a contradiction. Hence $t(X/X') = X/X'$ holds. Now we can prove that t is a preradical. To prove this first we show that if $t(X) = X$ and $t(Y) = 1$ then $Hom(X, Y) = 0$. Let $f : X \to Y$ be a morphism. Then $t(f(X)) = f(X)$. But since $\mathcal{F}$ is closed under subgroups, $f(X) \in \mathcal{F}$. Hence $f(X) = 1$. Next let $g : G' \to G$ be a morphism. Then since $Hom(t(G'), G/t(G)) = 0$, $g(t(G')) \subset t(G)$ holds. This means that t is a preradical. Finally we show that t is a radical. Since $Hom(t(G/t(G)), G/t(G)) = 0$, the inclusion map $t(G/t(G)) \hookrightarrow G/t(G)$ is null. Thus $t(G/t(G)) = 1$. Therefore we have proved that $\mathcal{F}$ is the torsion free class of the torsion theory associated with the idempotent radical t. This completes the proof.*

Corollary 2.2 *If t is an idempotent radical then $t(G)$ char G for all $G \in \mathcal{G}$.*

Example 2.1 *Let $\mathcal{S}$ be the class of finite solvable groups. Then it is well known that $\mathcal{S}$ is closed under subgroups, epimorphic images and group extensions. Hence $\mathcal{S}$ is a torsion free class of some torsion theory. There is a way to determine the corresponding idempotent radical. Let $D(G)$ be the commutator group of G. Then there exists an integer n depending on G such that $D^n(G) = D^m(G)$ for any $m \geq n$. So put $t(G) = D^n(G)$. Then t is the corresponding idempotent radical. By Proposition 1.4 t is epi-preserving.*

Example 2.2 *Let p be a prime number and $\mathcal{F}_p$ the class of all p-groups. Then $\mathcal{F}_p$ is closed under subgroups, epimorphic images and group extensions. Hence $\mathcal{F}_p$ is a torsion free class of some torsion theory.*

Finally let us consider the dual case of Theorem 2.1. If $\mathcal{T}$ is a torsion class of some torsion theory, then bye Theorem 1.1 $\mathcal{T}$ is closed under epimorphic images and group extensions. Does the converse hold? The answer is "no". The following example shows this.

Example 2.3 *Let $\mathcal{S}$ be the same as Example 2.1. Then $\mathcal{S}$ is not a torsion class of any torsion theory.*

Proof *Suppose $\mathcal{S}$ is a torsion class corresponding to an idempotent radical r. Since the alternative group A_4 is solvable, $r(A_4) = A_4$. But A_5 is simple and not solvable. This fact implies $r(A_5) = 1$. On the other hand $r(A_4) \to r(A_5)$ must be monomorphic. This is a contradiction. Hence $\mathcal{S}$ is not a torsion class of any torsion theory.*

References

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