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A Calculation of Rational Cohomology Classes on Some Complex Tori

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In this paper, we are concerned with complex tori T. For the rational cohomology group $H^{p,p}(T, Q)$ of type (p, p) on T, we have known its dimension, by example, in case of generic abelian varieties [1], n copies E^n of an elliptic curve E [2]. But in general, it is difficult to compute it. Here for the Cartesian product T of n elliptic curves $E_1, E_2, \cdots, E_n$, we shall compute the dimension of $H^{1,1}(T, Q)$. Furthermore we examine the dimension of $H^{2,2}(T, Q)$ for some compex tori T's.

§ 1. Definitions and notation

Throughout this paper, let C, Q be fields of complex, rational numbers, respectively. Let $T=V/L$ be a complex tous where V is a compex vector space of dim n and L is a lattice in V. We denote by $(z_1, z_2, \cdots, z_n)$, $(u_1, u_2, \cdots, u_{2n})$, the compex, real coordnates on V with respect to a fixed complex base of V and a fixed base of L, respectively. Then we have the following matrix A of size $(n, 2n)$, called a period matrix of T,

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{2n} \end{bmatrix}$$

and for differential 1-forms on T,

$$\begin{bmatrix} dz_1 \\ dz_2 \\ \vdots \\ dz_n \\ d\bar{z}_1 \\ \vdots \\ d\bar{z}_n \end{bmatrix} = \begin{bmatrix} A \\ \bar{A} \end{bmatrix} \begin{bmatrix} du_1 \\ du_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ du_{2n} \end{bmatrix}.$$

It is known that the complex de Rham cohomology group $H^{p,q}(T, C)$ of type (p, q) on T is generated by $dz_{i1} \wedge \cdots \wedge dz_{ip} \wedge d\bar{z}_{j1} \wedge \cdots \wedge d\bar{z}_{jq}$ with constant coefficients. We put $H^{p,p}(T, Q) = H^{p,p}(T, C) \cap H^{2p}(T, Q)$. For an element F of $H^{p,p}(T, C)$, we put

$$\begin{aligned}F&=\sum a_{ij}dz_{i1}\wedge\cdots\wedge dz_{ip}\wedge d\bar{z}_{j1}\wedge\cdots\wedge d\bar{z}_{jp}\\&=\sum b_k du_{k1}\wedge\cdots\wedge du_{k2p}\,.\end{aligned}$$

F is rational if and only if b_k's are rational numbers. We denote by $r_p=r_p(T)$ the dimension of $H^{p,p}(T,Q)$ over Q.

§ 2. $r_1(T)$ for $T=E_1\times E_2\times\cdots\times E_n$

Let E_i be elliptic curves with period matrices $(1, e_i)$, where $e_i\in C$, $i=1, 2, \cdots, n$. The cartesian product $T=E_1\times\cdots\times E_n$ of E_i's has a period matrix

$$A=\begin{bmatrix}1 & & & e_1 & & \\ & 1 & & & e_2 & \\ & & \ddots & & & \ddots \\ & & 1 & & & e_n\end{bmatrix}.$$

For an element of $F=\sum a_{ij}dz_i\wedge d\bar{z}_j$ of $H^{1,1}(T, C)$ we put

$$\begin{aligned}&F_i=a_{ii}dz_i\wedge d\bar{z}_i\,, \qquad (i=1, 2, \cdots, n)\\&F_{ij}=a_{ij}dz_i\wedge d\bar{z}_j+a_{ji}dz_j\wedge d\bar{z}_i\,, \qquad (1\leqq i<j\leqq n)\,.\end{aligned}$$

Then $F=F_1+\cdots+F_n+F_{12}+\cdots+F_{n-1,n}$. It is easy that F is rational if and only if F_i's and F_{ij}'s are rational. For F_i, F_{ij} are linearly independent over Q, we can know $r_1(T)$ by computing the numbers of rational F_i and rational F_{ij}. There exist n rational F_i and for $E_i\sim E_j$ (isogenous with complex multiplications) 2 rational F_{ij}, for $E_i\sim E_j$ (isogenous without complex multiplications) 1 rational F_{ij}, for $E_i\not\sim E_j$ (not isogenous) 0 rational F_{ij}. Therefore, by classifying $E_1, E_2, \cdots, E_n$ with respect to isogeny, we have the following theorem.

THEOREM. *Let $T=E_1^{i1}\times\cdots\times E_k^{ik}\times E_1'^{j1}\times\cdots\times E_h'^{jh}$ be the Cartesian product of elliptic curves where E_i's are of CM type and E_j''s are not of CM type ($E_i\not\sim E_j$, $E_i'\not\sim E_j'$ not isogenous). Then*

$$\dim H^{1,1}(T, Q)=n+2\binom{i_1}{2}+\cdots+2\binom{i_k}{2}+\binom{j_1}{2}+\cdots+\binom{j_h}{2}\,,$$

provided that $\binom{1}{2}=0$.

COROLLARY. *Let $T=E_1\times\cdots\times E_n$ be the Cartesian product of n elliptic curves E_i $(i=1, 2, \cdots, n)$. Then*

(1) $n\leqq r_1(T)\leqq n^2$.

(2) $r_1(T)=n\Leftrightarrow E_i\not\sim E_j$ *not isogenous for any* $i\neq j$.

(3) $r_1(T)=n^2\Leftrightarrow E_i\sim E_j$ *isogenous for any i, j and E_i is of CM type.*

For example, in case of $n=3, 4$, we have the following tables of $r_1=\dim H^{1,1}(T, Q)$.

$n=3$

CM type $i_1, \cdots, i_k$	not CM type $j_1, \cdots, j_h$	r_1
3	0	9
0	3	6
2, 1	0	5
2	1	5
1	2	4
0	2, 1	4
1, 1, 1		3

$n=4$

CM type $i_1, \cdots, i_k$	not CM type $j_1, \cdots, j_h$	r_1
4	0	16
0	4	8
3, 1	0	10
3	1	10
1	3	7
0	3, 1	7
2, 2	0	8
2	2	7
0	2, 2	6
2, 1, 1	0	6
2, 1	1	6
2	1, 1	6
1, 1	2	5
1	2, 1	5
0	2, 1, 1	5
1, 1, 1, 1		4

§ 3. $H^{2,2}(E_1\times E_2\times E_3\times E_4)$

If a complex torus T of dim n is an abelian variety, we have $r_p(T)=r_{n-p}(T)$ [3]. Hence we may compute $r_p(T)$ for $p=1, 2, \cdots, [n/2]$. Especially in case of $n=3$, we have $r_2(T)=r_1(T)$. In case of $n=4$, we have $r_3(T)=r_1(T)$, therefore we may examine $r_2(T)$.

Now we shall examine $r_2(T)$ in case of $T=E_1\times E_2\times E_3\times E_4$. We put $T'=E_1\times E_2\times E_3$. By Kunneth decomposition,

$$H^{2,2}(T)=[H^{0,0}(T')\wedge H^{2,2}(E_4)]_Q \qquad \cdots\cdots(^*1)$$
$$+[H^{1,0}(T')\wedge H^{1,2}(E_4)+H^{0,1}(T')\wedge H^{2,1}(E_4)]_Q \qquad \cdots\cdots(^*2)$$
$$+[H^{2,0}(T')\wedge H^{0,2}(E_4)+H^{1,1}(T')\wedge H^{1,1}(E_4)+H^{0,2}(T')\wedge H^{2,0}(E_4)]_Q \cdots\cdots(^*3)$$
$$+[H^{2,1}(T')\wedge H^{0,1}(E_4)+H^{1,2}(T')\wedge H^{1,0}(E_4)]_Q \qquad \cdots\cdots(^*4)$$
$$+[H^{2,2}(T')\wedge H^{0,0}(E_4)]_Q\,. \qquad \cdots\cdots(^*5)$$

For the terms (*1), $\cdots$, (*4) are linearly independent, we may compute the dimensions of each term. The terms (*1) and (*2) are equal to 0. The term (*3) is equal to $[H^{1,1}(T')\wedge H^{1,1}(E_4)]_Q$, hence isomorphic to $H^{1,1}(T', Q)$ and its dimension is known from the table of $n=3$. The term (*5) is isomorphic to $H^{2,2}(T', Q)$ and known as above. The term (*4) is decomposed to the following 7 independent parts.

(*4.1) $dz_1\wedge d\bar{z}_1\wedge d\bar{z}_2\wedge dz_4$ *and its conjugate*

(*4.2) $dz_1\wedge d\bar{z}_1\wedge d\bar{z}_3\wedge dz_4$ *and its conjugate*

(*4.3) $dz_2\wedge d\bar{z}_2\wedge d\bar{z}_1\wedge dz_4$ *and its conjugate*

(*4.4) $dz_2 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \wedge dz_4$ *and its conjugate*

(*4.5) $dz_3 \wedge d\bar{z}_3 \wedge d\bar{z}_1 \wedge dz_4$ *and its conjugate*

(*4.6) $dz_3 \wedge d\bar{z}_3 \wedge d\bar{z}_2 \wedge dz_4$ *and its conjugate*

(*4.7) $dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \wedge dz_4$, $d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_3 \wedge dz_4$, $d\bar{z}_1 \wedge d\bar{z}_2 \wedge dz_3 \wedge dz_4$, *and their conjugates.*

The dimension of (*4.1) is equal to $r_1(E_2 \times E_4)-2$. For (*4.2), $\cdots$, (*4.6), we can compute their dimensions as above. But the term (*4.7) has various situations. So we shall examine some examples.

EXAMPLES. Let E, E' be elliptic curves with period matrices $(1, i)$, $(1, pi)$, respectively. Let p be an irrational real number, then E is not isogenous to E'.

(1) In case of $T=E^3 \times E'$, we shall compute $r_2(T)$. The dimensions of (*3) and (*5) are equal to 9 from the table of $n=3$. The dimensions of (*4.1), $\cdots$, (*4.6) are equal to 0 [2]. The dimension of (*4.7) is computed directely and equal to 0. Hence we have $r_2(T)=18$.

(2) In case of $T=E^2 \times E'^2$, we have

dim (*3) = dim (*5) = 5

dim (*4.1) = dim (*4.3) = dim (*4.5) = dim (*4.6) = 0

$$\dim\,(^*4.2)=\dim\,(^*4.4)=\begin{cases}1 & \text{if } p^2 \text{ is irrational}\\ 2 & \text{if } p^2 \text{ is rational}\end{cases}$$

$$\dim\,(^*4.7)=\begin{cases}2 & \text{if } p^2 \text{ is irrational}\\ 4 & \text{if } p^2 \text{ is rational}.\end{cases}$$

Hence $r_2(T)=\begin{cases}14 & \text{if } p^2 \text{ is irrational}\\ 18 & \text{if } p^2 \text{ is rational}.\end{cases}$

References

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