Science Reports of the Faculty of Education, Gunma University. Vol. 32, pp. 1 — 6, 1983

A Note on Non-Simple Abelian Surfaces

Takeaki MURASAKI

Department of Mathematics, Faculty of Education, Gunma University
Maebashi, Gunma 371, Japan.
(Received Sept. 17, 1983)

Introduction

Let S be an abelian surface over the field C of complex numbers. Then we have $1 \leqq r(S) \leqq 4$ where $r(S) = \dim_{\mathbf{Q}} H^{1,1}(S, \mathbf{Q})$. If $r(S) = 1$, S is simple. If $r(S) = 4$, by Shioda [3] , S is non-simple and furthermore isomorphic to a direct product of elliptic curves.

In this paper we deal with the case of $r(S) = 2$ or 3. We shall propose some non-simple abelian surfaces which are never isomrophic to any direct product of elliptic curves.

§1. A rational representation of a homomorphism

Let $\mathbf{Q}$ be the field of rational numbers and $\mathbf{Z}$ be the module of integers. Let $T_1 = V_1/L_1$ and $T_1 = V_2/L_2$ be complex tori where V_1, V_2 are complex vector spaces and L_1, L_2 are these lattices. For an isogeny $f: T_1 \to T_2$, let $\hat{f}: V_1 \to V_2$ be the linear extension of f and $f_0 : L_1 \to L_2$ be its restriction to L_1.

LEMMA. *If $f: T_1 \to T_2$ is an isogeny, then card $(\ker f) = |\det f_0|$.*

PROOF. The $\hat{f}$ is a non-degenerate linear map. Hence

$$\ker f = \hat{f}^{-1}(L_2)/L_1 = L_2/f_0(L_1).$$

From the fundamental theorem of abelian groups, we have $\mathrm{card}(L_2/f_0(L_1)) = |\det f_0|$.

For a complex torus $T = V/L$ of dim n, choosing a suitable base $(\mathbf{e}_1, \cdots, \mathbf{e}_n, \mathbf{g}_1, \cdots, \mathbf{g}_n)$ of L, we have a period matrix of T as the following form,

$$(E, G) = \begin{pmatrix} 1 & & & & g_{11} & \cdots\cdots & g_{1n} \\ & 1 & & & & \cdots\cdots\cdots & \\ & & \ddots & & & & \\ & & & 1 & g_{n1} & \cdots\cdots & g_{nn} \end{pmatrix}, \ \det(G - \bar{G}) \neq 0.$$

It is called a normalized period matrix of T.

PROPOSITION 1. *Let T_i be a complex torus of dim n_i with a normalized period* *matrix* (E, G_i). $(i=1, 2)$ *A matrix* F *in* $\mathbf{M}(2n_2\times 2n_1, \mathbf{Z})$ *is a rational representation of a homomorphism* $f: T_1 \to T_2$, *if and only if*

$$(E, G_2)\ F\begin{pmatrix} G_1 \\ -E \end{pmatrix} = O.$$

PROOF. Let f be a homomorphism of T_1 to T_2. Then for its analytic representation H and its rational representation F, we have

$$H\ (E, G_1) = (E, G_2)\ F.$$

Hence

$$(E, G_2)\ F\begin{pmatrix} G_1 \\ -E \end{pmatrix} = H(E, G_1)\begin{pmatrix} G_1 \\ -E \end{pmatrix} = O.$$

Inversely, let F be a matrix in $\mathbf{M}(2n_2\times 2n_1, \mathbf{Z})$ satisfying the condition. Set $F=(F_1, F_2)$ where F_i is one in $\mathbf{M}(2n_2\times n_1, \mathbf{Z})$. If we put

$$(E, G_2)\ F_1 = H \in \mathbf{M}(n_2\times n_1, \mathbf{C})$$
$$(E, G_2)\ F_2 = K \in \mathbf{M}(n_2\times n_1, \mathbf{C}),$$

then we have $K = HG_1$ from the above condition. Hence

$$\begin{aligned} H\ (E, G_1) &= (E, G_2)\ F_1\ (E, G_1) = ((E, G_2)\ F_1, (E, G_2)\ F_1\ G_1) \\ &= ((E, G_2)\ F_1, HG_1) = ((E, G_2)\ F_1, K) \\ &= ((E, G_2)\ F_1, (E, G_2)\ F_2) = (E, G_2)\ F. \end{aligned}$$

This means that there exists a homomorphism with F as its rational representation.

PROPOSITION 2. *Let* T *be a complex torus of dim* n *whih a normalized period matrix* (E, G). *A matrix* F *in* $\mathbf{M}(2n\times 2, \mathbf{Z})$ *is a rational representaion of a non-zero homomorphism* $f: E \to T$ *where* E *is an elliptic curve, if and only if* F *satisfies the following conditions*

(1) *rank of* $F = 2$

(2) *rank of* $(E, G)\ F = 1$.

PROOF. If F is a rational representation of $f: E \to T$, by PROPOSITION 1, it is obvious that F satisfies these conditions. Inversely, let F satisfy (1), (2). From (2), there exists a non-zero vector (a, b) such that

$$(E, G)\ F\begin{pmatrix} a \\ b \end{pmatrix} = O.$$

Furthermore, from (1), we have $a \neq 0$ and $b \neq 0$. Hence we may put $a = g$ and $b = -1$. On the other hand, from (1), we see that g is not real. Set an elliptic curve $E = \mathbf{C}/(1, g)$, then, from PROPOSITION 1, we have the homomorphism $f : E \to T$ determined by F.

The condition (2) in PROPOSITION 2 is equivalent to that $\det((E, G)F) = 0$, because $\det(G-\bar{G}) \neq 0$.

§2. Non-simple abelian surfaces

Let T be a comlex torus of dim 2. We fix a normalized period matrix (E, G) of T. Then we put

$$M(G) = \{A \in \mathbf{M}(4\times 2, \mathbf{Z}) \mid (1)\ \text{rank of } A = 2,\ (2)\ \det((E, G)A) = 0\}$$

$$\begin{aligned} M^2(G) &= M(G) \times M(G) \\ &= \{(A, B) \mid A, B \in M(G)\} \subset \mathbf{M}(4\times 4, \mathbf{Z}). \end{aligned}$$

By PROPOSITION 2, T is non-simple iff $M^2(G) \neq \phi$.

PROPOSITION 3. *Let T be a complex torus of dim 2 with a normalized period matrix (E, G). Then,*

(1) T is isogenous to a direct product of two elliptic curves if and only if there exists an element C in $M^2(G)$ such that $\det C \neq 0$.

(2) T is isomorphic to a direct product of two elliptic curves if and only if there exists an element C in $M^2(G)$ such that $\det C = 1$.

PROOF. (1) Let $E_i = \mathbf{C}/(1, g_i)$ be an elliptic curve. $(i = 1, 2)$ If $f : E_1 \times E_2 \to T$ is an isogeny, from PROPOSITION 1, we have a matrix F in $\mathbf{M}(4\times 4, \mathbf{Z})$ such that

$$(E, G)\, F \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} = 0 \text{ and } \det F \neq 0.$$

Let $F = (x_1, x_2, x_3, x_4)$ where x_i are 4-vectors, then we have

$$(E, G)\,(x_1, x_3) \begin{pmatrix} g_1 \\ -1 \end{pmatrix} = 0, \quad (E, G)\,(x_2, x_4) \begin{pmatrix} g_2 \\ -1 \end{pmatrix} = 0.$$

Therefore if we put $A = (x_1, x_3)$, $B = (x_2, x_4)$, then we know that $C = (A, B)$ is an element in $M_2(G)$ such that $\det C \neq 0$.

Inversely, let $C = (y_1, y_2, y_3, y_4)$ in $M^2(G)$ and $\det C \neq 0$, then we have $A = (y_1, y_3)$ and $B =$

(y_2, y_4) in $M(G)$. Hence from PROPOSITION 2, we have two homomorphisms f and f' to T determined by A and B.

Now $f \times f'$ is an isogeny asked for.

(2) It is proved easily by LEMMA.

For a non-simple abelian surface S, we define

$$m(S) = \min \{ |\det C| \mid C \in M^2(G),\ C \text{ is regular} \}.$$

From LEMMA, the value $m(S)$ is uniquely detemined by S, independent of a normalized period matrix of S. S is isomorphic to a direct product of elliptic curves iff $m(S) = 1$. In the following §, for $r(S) = 2, 3$, we shall give examples for which $m(S) = 2$.

§3. Construction of examples

Let $G = \begin{pmatrix} a & a \\ -b & b \end{pmatrix}$ be a matrix in $\mathbf{M}(2\times 2, \mathbf{C})$ such that $\det(G-\bar{G}) \neq 0$. Let S be a complex torus of dim 2 with a normalized period matrix (E, G). Then S is isogenous to a product $E_1 \times E_2$ where $E_1 = \mathbf{C}/(1, 2a)$ and $E_2 = \mathbf{C}/(1, 2b)$. Hence S is algebraic.

(I) An example in $r(S) = 2$

Let $\{1, a, b, ab\}$ assume linearly independent over $\mathbf{Q}$. (e. g. $a=i$, $b=ei$) Then we know $r(S) = 2$ from COROLLARY 2 in [2]. Now we shall prove $m(S) = 2$.

Let $A = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ be an element of $M(G)$ where u_i' s are integral 2-vectors. From $\det((E, G)A) = 0$, we have

$$\begin{vmatrix} u_1 \\ u_2 \end{vmatrix} + \begin{vmatrix} u_2+u_4 \\ u_2 \end{vmatrix} a + \begin{vmatrix} u_1 \\ -u_3+u_4 \end{vmatrix} b + 2\begin{vmatrix} u_3 \\ u_4 \end{vmatrix} ab = 0.$$

By the assumption, we have

(1) $\begin{vmatrix} u_1 \\ u_2 \end{vmatrix} = 0$

(2) $\begin{vmatrix} u_3 \\ u_2 \end{vmatrix} + \begin{vmatrix} u_4 \\ u_2 \end{vmatrix} = 0$

(3) $-\begin{vmatrix} u_1 \\ u_3 \end{vmatrix} + \begin{vmatrix} u_1 \\ u_4 \end{vmatrix} = 0$

(4) $\begin{vmatrix} u_3 \\ u_4 \end{vmatrix} = 0.$

From the condition that rank of $A = 2$, let $\{u_1, u_3\}$ suppose linearly independent over $\mathbf{Q}$.

Then from (1), (4), we have $u_2 = pu_1$, $u_4 = qu_3$ where p, q rational. From (2), (3), we have $p=0$, $q=1$. Hence we see that $A = \begin{pmatrix} u \\ 0 \\ v \\ v \end{pmatrix}$ where $\{u, v\}$ are linearly independent over **Q.** If $\{u_1, u_4\}$ are linearly independent over **Q,** then A is also the above form. If $\{u_2, u_3\}$ are linearly independent over **Q,** then $A = \begin{pmatrix} 0 \\ u \\ v \\ -v \end{pmatrix}$ where $\{u, v\}$ are linearly independent over **Q.** If $\{u_2, u_4\}$ are linearly indepedent over **Q,** A is also the above form. Therefore, if det $F \neq 0$ for an element F in $M^2(G)$, then $F = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \\ v_1 & v_2 \\ v_1 & -v_2 \end{pmatrix}$

Hence we have det $F = 2 \begin{vmatrix} u_1 \\ v_1 \end{vmatrix} \begin{vmatrix} u_2 \\ -v_2 \end{vmatrix}$ and m $(S) = 2$.

(II) An example in $r(S) = 3$

Let $\{1, a, b\}$ be linearly independent over **Q,** and assume that det $G = 2ab = n$ is an integer $\neq 0$. (e. g. $a = ei$, $b = 1/ei$) Then we know $r(S) = 3$ from COROLLARY 2 in [2]. Now we shall prove $m(S) = 2$.

Let $A = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ be an element of $M(G)$. From det $((E, G)A) = 0$, we have

$$\left(\begin{vmatrix} u_1 \\ u_2 \end{vmatrix} + n \begin{vmatrix} u_3 \\ u_4 \end{vmatrix} \right) + \begin{vmatrix} u_3 + u_4 \\ u_2 \end{vmatrix} a + \begin{vmatrix} u_1 \\ -u_3 + u_4 \end{vmatrix} b = 0.$$

Hence we have

(1) $\begin{vmatrix} u_1 \\ u_2 \end{vmatrix} + n \begin{vmatrix} u_3 \\ u_4 \end{vmatrix} = 0$

(2) $\begin{vmatrix} u_3 + u_4 \\ u_2 \end{vmatrix} = 0$

(3) $\begin{vmatrix} u_1 \\ -u_3 + u_4 \end{vmatrix} = 0$

If $\{u_1, u_2\}$ are linearly dependent over **Q,** then, from (1), $\{u_3, u_4\}$ are also linearly dependent over **Q** and it reduces to the above case (I).

Hence $A = \begin{pmatrix} u \\ 0 \\ v \\ v \end{pmatrix}$ or $\begin{pmatrix} 0 \\ u \\ v \\ -v \end{pmatrix}$, which is called of type (I).

If $\{u_1, u_2\}$ are linearly independent over **Q,** then $\{u_3, u_4\}$ are also linealy independent over **Q.** Then from (2), (3), we have $u_1 = p(-u_3 + u_4)$, $u_2 = q(u_3 + u_4)$ where p, q are rational.

Therefore we know $A = \begin{pmatrix} p\,(-u+v) \\ q\,(\;\;u+v) \\ u \\ v \end{pmatrix}$ where u, v are linearly independent integral 2-vectors and, from (1), $n = 2pq$. This A is called of type (II).

Now consider a regular matrix $C = (A, B)$ in $M^2(G)$.

(*) The case that A and B are of type (I).

It reduces to the previous case (I), therefore det C is even.

(**) The case that A is of type (I) and B is of type (II).

$$\det C = \begin{vmatrix} u & u_1 \\ 0 & u_2 \\ v & u_3 \\ v & u_4 \end{vmatrix} = \begin{vmatrix} u & 0 \\ -2v & 0 \\ v & u_3 \\ v & u_4 \end{vmatrix} = 2 \begin{vmatrix} u \\ -v \end{vmatrix} \begin{vmatrix} u_3 \\ u_4 \end{vmatrix}.$$ Hence det C is even.

(***) The case that A and B are of type (II).

$$\det C = \begin{vmatrix} -p & p & -p' & p' \\ q & q & q' & q' \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} \begin{vmatrix} u_3 & 0 \\ u_4 & 0 \\ 0 & u'_3 \\ 0 & u'_4 \end{vmatrix} = 2(p'-p)(q-q') \begin{vmatrix} u_3 \\ u_4 \end{vmatrix} \begin{vmatrix} u_3' \\ u_4' \end{vmatrix}$$

$$= -(2pq+2p'q') \begin{vmatrix} u_3 \\ u_4 \end{vmatrix} \begin{vmatrix} u_3' \\ u_4' \end{vmatrix} + 2\,p'q' \begin{vmatrix} u_3 \\ u_4 \end{vmatrix} \begin{vmatrix} u_3' \\ u_4' \end{vmatrix} + 2pq' \begin{vmatrix} u_3 \\ u_4 \end{vmatrix} \begin{vmatrix} u_3' \\ u_4' \end{vmatrix}$$

$$= -2n \begin{vmatrix} u_3 \\ u_4 \end{vmatrix} \begin{vmatrix} u_3' \\ u_4' \end{vmatrix} - 2 \begin{vmatrix} u_3 \\ u_2 \end{vmatrix} \begin{vmatrix} u_1' \\ u_4' \end{vmatrix} - 2 \begin{vmatrix} u_1 \\ u_4 \end{vmatrix} \begin{vmatrix} u_3' \\ u_2' \end{vmatrix}.$$

Hence det C is even.

Therefore, in this case (II), we have $m(S) = 2$.

References

1) Koizumi, S. *Theta function. (Japanese) Kinokuniya, 1982.*

2) Murasaki, T. A classification of complex tori. Gunma Journal of Liberal Arts and Science, Gunma Univ. vol. 11, no. 7, pp. 103-108, 1977.

3) Shioda, T. and Mitani, N. Singular abelian surfaces and binary quadratic forms. Lecture Notes in Mathematics, No.412, pp. 259-287, 1974. Springer.

4) Weil, A *Variétés kahlériennes.* Hermann, 1957.